Supercurrent in superconducting graphene

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The problem of supercurrent in superconducting graphene is revisited and the supercurrent is calculated within the mean-field model employing the two-component wave functions on a honeycomb lattice with pairing between different valleys in the Brillouin zone. We show that the supercurrent within the linear approximation in the order-parameter-phase gradient is always finite even if the doping level is exactly zero.

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I. INTRODUCTION

Recent exciting developments in transport experiments on graphene[1] have stimulated theoretical and experimental studies of possible superconductivity phenomena in this material. Experimentally, there are both hints toward intrinsic superconductivity[2] and observations of proximity-induced superconductivity in graphene layers.[3,4] Intrinsic superconductivity has been discussed theoretically in the frameworks of phonon- and plasmon-mediated mechanisms[5,6] whereas resonating valence-bond and density-wave lattice models were proposed in Refs. 7–9. It was shown within the BCS model[8,10,11] that the superconducting transition in the undoped graphene possesses a quantum critical point at a finite interaction strength below which the critical temperature vanishes. However, electrons in graphene may become unstable toward formation of Cooper pairs for any finite pairing interaction if doping shifts the Fermi level by an amount μ away from the Dirac point.[5,8,9,11] The effect of fluctuations on the critical temperature of superconducting transition in graphene has been studied in Ref. 12. A number of unusual features of superconducting state have been predicted, which are closely related to the Dirac-type spectrum of normal-state excitations. In particular, the unconventional normal electron dispersion has been shown to result in a nontrivial modification of Andreev reflection[13] and Andreev bound states in Josephson junctions[14] and vortex cores (see Ref. 15, and references therein).

Nevertheless, there still remains a controversy regarding the most fundamental property of superconducting graphene, i.e., the *supercurrent*, no matter what the mechanism, intrinsic or extrinsic, of the superconductivity is. In Ref. 8 the supercurrent has been calculated within the framework of the mean-field model of superconducting graphene[8,13] that assumes the Cooper pairing between electrons belonging to the same sublattice in the configurational space. According to Ref. 8 the supercurrent calculated as a linear response to the phase gradient of the order parameter disappears in undoped graphene (i.e., zero shift of the chemical potential, $\mu=0$) at zero temperature even if the order parameter Δ itself is finite. However, a simpler model based on an effective Dirac-type spectrum of normal electrons[11] demonstrates that the supercurrent is always finite as long as superconductivity exists, $\Delta\neq 0$. In fact, a nonzero supercurrent at the Dirac point was seen experimentally for proximity-induced superconductivity in graphene.[3,4] Though the surprising result[8] of "superconductivity without supercurrent" is an alarming indication by itself, the question may be raised, to which extent this difference between the supercurrents is model dependent,[16,17] or, if not, what is then the correct behavior of the supercurrent in the low-doping limit, $\mu\to 0$.

In the present paper we revisit this problem and calculate the supercurrent again using the two-component mean-field model of superconductivity in graphene as formulated in Refs. 8 and 13. Performing explicit calculations we show that the supercurrent is indeed *always finite*. Its value in the low-doping limit $\mu\ll|\Delta|$ is independent of whether the doping level is exactly zero or not, in contrast to the claim of Ref. 8. This statement qualitatively agrees with the conclusion drawn from the simple model suggested in Ref. 11.

The paper is organized as follows. In the next section we outline the model of superconductivity in graphene as formulated in Refs. 8 and 13 and introduce the basic quantities relevant for further calculations. In Sec. III we calculate the supercurrent within the linear approximation in the order-parameter-phase gradient for finite doping levels. The last Sec. III B deals with the case of low doping $\mu\ll|\Delta|$. Details of calculations are presented in Appendices A and B.

II. BOGOLIUBOV-DE GENNES-DIRAC EQUATIONS

Transport properties of graphene associated with energies much smaller than the bandwidth are conveniently described by equations of the Dirac type for two-component wave function whose two components are envelopes of the true wave functions for two sublattices in the configurational space, Fig. 1(a), near the so-called Dirac points $\mathbf{K}$ or $\mathbf{K}'$ in the Brillouin zone of the reciprocal lattice, Fig. 1(b) (for more details see, for example, Refs. 13 and 18).

A holelike excitation $\Psi_{\mathbf{K}}^{(h)}$ in the valley associated with the point $\mathbf{K}$ is the complex conjugated wave function of a particlelike excitation in the valley $-\mathbf{K}$, i.e., $\Psi_{\mathbf{K}}^{(h)}=\Psi_{-\mathbf{K}}^{*}$. In what follows we denote particlelike states by u while holelike states by v. The Bogoliubov-de Gennes equations have the form[8,13]

$$v_F\boldsymbol{\sigma}\cdot\left(-i\nabla-\frac{e}{c}\mathbf{A}\right)\hat{u}(\mathbf{r})+\Delta\hat{v}(\mathbf{r})=(\epsilon+\mu)\hat{u}(\mathbf{r}), \qquad (1)$$

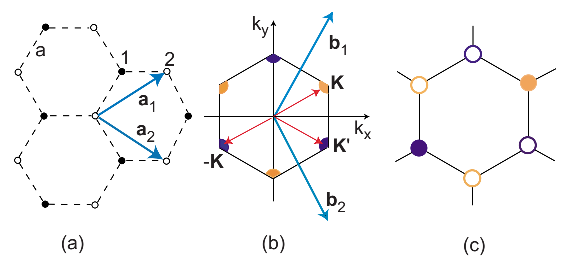

FIG. 1. (Color online) (a) Unit cell with two sublattices 1 (black dots) and 2 (open dots), interatomic distance a, and the basis vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. (b) Brillouin zone with the reciprocal-lattice vectors $\mathbf{b}_1$ and $\mathbf{b}_2$. $\mathbf{K}$ and $\mathbf{K}'$ show the two nonequivalent Dirac corners (differently shaded sectors) of the Brillouin zone; other corners are obtained by shifting these two by integer linear combinations $n_1\mathbf{b}_1+n_2\mathbf{b}_2$. (c) Dirac cone regions (circles) in the extended zone scheme. Filled circles belong to the same zone while open circles are from other zones.

$$-v_F\boldsymbol{\sigma}\cdot\left(-i\nabla+\frac{e}{c}\mathbf{A}\right)\hat{v}(\mathbf{r})+\Delta^*\hat{u}(\mathbf{r})=(\epsilon-\mu)\hat{v}(\mathbf{r}). \quad (2)$$

The two-component wave functions are in a form of pseudospinors,

$$\hat{u}=\begin{pmatrix}u_1\\u_2\end{pmatrix},\quad \hat{v}=\begin{pmatrix}v_1\\v_2\end{pmatrix},\quad \hat{u}^\dagger=(u_1^*,u_2^*),\quad \hat{v}^\dagger=(v_1^*,v_2^*),$$

where the two components are the wave functions of electrons and holes on two sublattices 1 and 2 in the honeycomb lattice, Fig. 1(a); $\hat{\boldsymbol{\sigma}}=(\hat{\sigma}_x,\hat{\sigma}_y)$ are Pauli matrices in the pseudospin space,

$$\hat{\sigma}_x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \hat{\sigma}_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}.$$

Equations for the valley at the point $\mathbf{K}'$ can be obtained with the replacement $u_1\to u_2$, $v_1\to v_2$.

Pairing of a particle u in the valley $\mathbf{K}$ in the Brillouin zone occurs with a particle in the valley $-\mathbf{K}$, i.e., with a hole v at $\mathbf{K}$. Since the points $-\mathbf{K}$ and $\mathbf{K}'$ are equivalent, $\mathbf{K}+\mathbf{K}'=\mathbf{b}_1+\mathbf{b}_2$, where $\mathbf{b}_1$ and $\mathbf{b}_2$ are the vectors of the reciprocal lattice, Fig. 1(b), one may also say that pairing is between particles from the valleys $\mathbf{K}$ and $\mathbf{K}'$. The model assumes that the order parameter is the same for both sublattices,

$$\Delta=-V\sum_{\mathbf{p},\alpha}(1-2f_{\mathbf{p},\alpha})\hat{v}^\dagger_{\mathbf{p},\alpha}\hat{u}_{\mathbf{p},\alpha}, \quad (3)$$

where α labels four independent solutions of the Bogoliubov-de Gennes equations with the momentum $\mathbf{p}$ (see below) and $f_{\mathbf{p},\alpha}$ is the Fermi occupation number in the state $\mathbf{p}$, α. The sum runs over all states within the Brillouin zone. We do not concentrate here on the specific nature of the pairing interaction assuming that the pairing potential may be either due to some intrinsic mechanism or due to an interaction induced by a proximity to a usual superconductor.

The particle density is

$$N=2\sum_{\mathbf{p},\alpha}[f_{\mathbf{p},\alpha}\hat{u}^\dagger_{\mathbf{p},\alpha}\hat{u}_{\mathbf{p},\alpha}+(1-f_{\mathbf{p},\alpha})\hat{v}^\dagger_{\mathbf{p},\alpha}\hat{v}_{\mathbf{p},\alpha}].$$

Factor 2 accounts for the true spin of electrons. The statistical average of the current operator is

$$\mathbf{j}=2ev_F\sum_{\mathbf{p},\alpha}[\hat{u}^\dagger_{\mathbf{p},\alpha}\hat{\boldsymbol{\sigma}}\hat{u}_{\mathbf{p},\alpha}f_{\mathbf{p},\alpha}-\hat{v}^\dagger_{\mathbf{p},\alpha}\hat{\boldsymbol{\sigma}}\hat{v}_\alpha(1-f_{\mathbf{p},\alpha})]. \quad (4)$$

Sometimes the currents $\mathbf{j}_e$ and $\mathbf{j}_p$ are defined,

$$\mathbf{j}_e=-ev_F\sum_{\mathbf{p},\alpha}[\hat{u}^\dagger_{\mathbf{p},\alpha}\hat{\boldsymbol{\sigma}}\hat{u}_{\mathbf{p},\alpha}+\hat{v}^\dagger_{\mathbf{p},\alpha}\hat{\boldsymbol{\sigma}}\hat{v}_\alpha](1-2f_{\mathbf{p},\alpha}), \quad (5)$$

$$\mathbf{j}_p=ev_F\sum_{\mathbf{p},\alpha}[\hat{u}^\dagger_\alpha\hat{\boldsymbol{\sigma}}\hat{u}_\alpha-\hat{v}^\dagger_\alpha\hat{\boldsymbol{\sigma}}\hat{v}_\alpha], \quad (6)$$

such that $\mathbf{j}=\mathbf{j}_e+\mathbf{j}_p$. The current $\mathbf{j}_p$ is the quasiparticle flux, which vanishes in our spatially uniform case (see below). The current $\mathbf{j}_e$ is the sum of currents in each state, which may be not conserved separately in some spatially inhomogeneous or nonequilibrium situations, but the total current, however, is conserved ($\mathrm{div}\,\mathbf{j}_e=0$) taking into account the self-consistency equation.[19]

We will consider the case of zero magnetic field and look for the solution in the form of plane waves,

$$\hat{u}_{\mathbf{p}}=\hat{u}e^{i(\mathbf{p}+\mathbf{k}/2)\cdot\mathbf{r}},\quad \hat{v}_{\mathbf{p}}=\hat{v}e^{i(\mathbf{p}-\mathbf{k}/2)\cdot\mathbf{r}}, \quad (7)$$

assuming that the order parameter $\Delta=|\Delta|e^{i\mathbf{k}\cdot\mathbf{r}}$ corresponds to a moving condensate of Cooper pairs. Equations (1) and (2) give[13]

$$v_F\hat{\boldsymbol{\sigma}}\cdot(\mathbf{p}+\mathbf{k}/2)\hat{u}+\Delta\hat{v}=(E+\mu)\hat{u}, \quad (8)$$

$$-v_F\hat{\boldsymbol{\sigma}}\cdot(\mathbf{p}-\mathbf{k}/2)\hat{v}+\Delta^*\hat{u}=(E-\mu)\hat{v}. \quad (9)$$

Ground state

Let us consider the ground state with zero current ($\mathbf{k}=0$) and also outline some known results[8,13,18] which will be used in what follows. Equations (8) and (9) define four linearly independent solutions. Let us introduce the spinors,

$$\hat{a}_\uparrow=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\dfrac{p_x-ip_y}{p}}\\\sqrt{\dfrac{p_x+ip_y}{p}}\end{pmatrix},\quad \hat{a}_\downarrow=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\dfrac{p_x-ip_y}{p}}\\-\sqrt{\dfrac{p_x+ip_y}{p}}\end{pmatrix}, \quad (10)$$

which satisfy

$$(\hat{\boldsymbol{\sigma}}\cdot\mathbf{p})\hat{a}_{\uparrow,\downarrow}=\pm p\hat{a}_{\uparrow,\downarrow}. \quad (11)$$

The spinors $\hat{a}_\uparrow$ and $\hat{a}_\downarrow$ are eigenstates of excitations in the normal graphene. We also introduce vectors in the Nambu space,

$$\check{\psi}=\begin{pmatrix}\hat{u}\\\hat{v}\end{pmatrix},\quad \check{\psi}^+=(\hat{u}^\dagger,\hat{v}^\dagger).$$

Each component here is a pseudospinor. We find for the upper sign in Eq. (11),

$$E_{1,2}^{(0)} = \pm E_\uparrow, \quad E_\uparrow = \sqrt{(v_F p - \mu)^2 + |\Delta|^2}. \tag{12}$$

For $\mathbf{k}=0$ the order parameter is real $\Delta=|\Delta|$. Therefore,

$$\begin{pmatrix} \hat{u}_1^{(0)} \\ \hat{v}_1^{(0)} \end{pmatrix} = \begin{pmatrix} u_\uparrow \\ v_\uparrow \end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p}\cdot\mathbf{r}}, \quad \begin{pmatrix} \hat{u}_2^{(0)} \\ \hat{v}_2^{(0)} \end{pmatrix} = \begin{pmatrix} v_\uparrow \\ -u_\uparrow \end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{13}$$

For the lower sign in Eq. (11) we have

$$E_{3,4}^{(0)} = \pm E_\downarrow, \quad E_\downarrow = \sqrt{(v_F p + \mu)^2 + |\Delta|^2} \tag{14}$$

and

$$\begin{pmatrix} \hat{u}_3^{(0)} \\ \hat{v}_3^{(0)} \end{pmatrix} = \begin{pmatrix} u_\downarrow \\ v_\downarrow \end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p}\cdot\mathbf{r}}, \quad \begin{pmatrix} \hat{u}_4^{(0)} \\ \hat{v}_4^{(0)} \end{pmatrix} = \begin{pmatrix} v_\downarrow \\ -u_\downarrow \end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p}\cdot\mathbf{r}}. \tag{15}$$

Here

$$u_\uparrow = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{v_F p - \mu}{E_\uparrow}}, \quad v_\uparrow = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{v_F p - \mu}{E_\uparrow}}, \tag{16}$$

$$u_\downarrow = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{v_F p + \mu}{E_\downarrow}}, \quad v_\downarrow = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{v_F p + \mu}{E_\downarrow}}. \tag{17}$$

The different wave functions are orthogonal, $\check{\psi}_\alpha^+ \check{\psi}_\beta = \delta_{\alpha\beta}$. Equation (13) goes over into Eq. (15) under the transformation $E \to -E$ and $\mu \to -\mu$. Using Eq. (10) one can check that

$$\hat{a}_\uparrow^\dagger \hat{\boldsymbol{\sigma}} \hat{a}_\uparrow = -\hat{a}_\downarrow^\dagger \hat{\boldsymbol{\sigma}} \hat{a}_\downarrow = \mathbf{p}/p, \tag{18}$$

$$\hat{a}_\uparrow^\dagger \hat{\boldsymbol{\sigma}} \hat{a}_\downarrow = -\hat{a}_\downarrow^\dagger \hat{\boldsymbol{\sigma}} \hat{a}_\uparrow = i[\mathbf{z}_0 \times \mathbf{p}]/p. \tag{19}$$

where $\mathbf{z}_0$ is the unit vector in the z direction perpendicular to the graphene layer plane.

III. CURRENT-CARRYING STATE

The set of linear Bogoliubov-de Gennes, Eqs. (8) and (9), has nontrivial solutions if the corresponding determinant is zero. For the current-carrying state $\mathbf{k} \neq 0$, the solvability condition takes the form

$$(E^2 - \mu^2)^2 - 2|\Delta|^2(E^2 - \mu^2) + |\Delta|^4 + 2|\Delta|^2 v_F^2 \mathbf{p}_+ \mathbf{p}_- - (E + \mu)^2 v_F^2 \mathbf{p}_-^2 - (E - \mu)^2 v_F^2 \mathbf{p}_+^2 + v_F^4 \mathbf{p}_+^2 \mathbf{p}_-^2 = 0, \tag{20}$$

where $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$.

Equation (20) cannot be solved analytically for nonzero $\mathbf{k}$, except for the zero-doping limit $\mu=0$. In the latter case the solvability condition, Eq. (20), becomes biquadratic and yields the energy spectrum,[16]

$$E_\pm^2 = |\Delta|^2 + v_F^2(p^2 + k^2/4) \pm \sqrt{|\Delta|^2 v_F^2 k^2 + v_F^4 (\mathbf{p}\cdot\mathbf{k})^2}. \tag{21}$$

In this limit, the Bogoliubov-de Gennes equations can also be solved analytically (see Appendix B 2). One sees that the energy, Eq. (21), for $\mu=0$ does not have the usual Doppler term proportional to the vector $\mathbf{k}$. This may lead to a confusion[16] when calculating the supercurrent.

A. Finite-doping linear response

Let us consider the linear correction to the energy and to the wave functions due to superconducting momentum $\mathbf{k}$ assuming $v_F|\mathbf{k}| \ll \mu$. We put $E = E(0) + E'$, where $E' \ll E(0)$ and $E(0)$ is the energy of one of the states with $\mathbf{k}=0$ determined by Eqs. (12) and (14). Within the linear approximation in $\mathbf{k}$ we find from Eq. (20) for any finite $\mu \neq 0$,

$$E' = \pm v_F(\mathbf{p}\cdot\mathbf{k})/2p \equiv \pm E_D$$

for the upper (lower) sign in Eq. (11). Therefore, corrections to the energies are

$$E_{1,2}^{(1)} = -E_{3,4}^{(1)} = E_D. \tag{22}$$

The energy $E_D = (d\xi_\mathbf{p}/dp)(\mathbf{k}/2)$ is the usual Doppler shift for the normal-state energy $\xi_\mathbf{p} = v_F p$. Equation (22) coincides with the result of Ref. 11 obtained in the linear approximation in $\mathbf{k}$. At the same time, it differs from the linear in $\mathbf{k}$ term obtained from Eq. (21) for $\mu=0$. This means that the undoped case $\mu=0$ requires a special consideration. This will be done later in Sec. III B (see also Appendix B).

First-order corrections to the wave functions can be found by expanding the total functions in terms of the zero-order functions $\hat{u}_\beta^{(0)}$, $\hat{v}_\beta^{(0)}$ given by Eqs. (13) and (15),

$$\check{\psi}_\alpha = \check{\psi}_\alpha^{(0)} + \sum_{\beta\neq\alpha} B_{\alpha\beta} \check{\psi}_\beta^{(0)}. \tag{23}$$

Inserting this into Eqs. (8) and (9) we find

$$B_{\alpha\beta} = \frac{v_F \check{\psi}_\beta^{(0)+} (\hat{\boldsymbol{\sigma}}\cdot\mathbf{k}) \check{\psi}_\alpha^{(0)}}{2(E_\alpha^{(0)} - E_\beta^{(0)})}.$$

One can check that $B_{\beta\alpha} = -B_{\alpha\beta}^*$. We find $B_{12} = B_{21} = B_{34} = B_{43} = 0$ while

$$B_{13} = -B_{24} = -\frac{iv_F([\mathbf{p}\times\mathbf{k}]\cdot\mathbf{z})}{2p} \frac{(u_\downarrow^* u_\uparrow + v_\downarrow^* v_\uparrow)}{E_\uparrow - E_\downarrow}, \tag{24}$$

$$B_{23} = B_{14} = \frac{iv_F([\mathbf{p}\times\mathbf{k}]\cdot\mathbf{z})}{2p} \frac{(u_\downarrow^* v_\uparrow - v_\downarrow^* u_\uparrow)}{E_\uparrow + E_\downarrow}. \tag{25}$$

Therefore, the up-spin wave functions $\hat{u}^{(1,2)}$ contain only corrections with the down-spin components $\hat{u}^{(3,4)}$, and vice versa. Expansion, Eq. (23), yields also the corrections to the eigenenergies which coincide with Eq. (22).

Using Eq. (23) one can show that the quasiparticle current $\mathbf{j}_p$ is zero. The supercurrent Eq. (4) takes the form of Eq. (5), $\mathbf{j} = \mathbf{j}_e$, which can be written as

$$\mathbf{j} = \int \frac{d^2p}{(2\pi)^2} [\mathbf{j}_\mathbf{K}(\mathbf{p}) + \mathbf{j}_{-\mathbf{K}}(\mathbf{p})], \tag{26}$$

where

$$\mathbf{j}_\mathbf{K}(\mathbf{p}) = -ev_F \sum_{\alpha=1}^{4} \hat{u}_{\mathbf{p},\alpha}^\dagger \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p},\alpha} [1 - 2f_{\mathbf{p},\alpha}], \tag{27}$$

$$\mathbf{j}_{-\mathbf{K}}(\mathbf{p}) = -ev_F\sum_{\alpha=1}^{4}\hat{v}^{\dagger}_{\mathbf{p},\alpha}\hat{\boldsymbol{\sigma}}\hat{v}_{\mathbf{p},\alpha}[1-2f_{\mathbf{p},\alpha}]. \tag{28}$$

The term $\mathbf{j}_{\mathbf{K}}(\mathbf{p})$ is the contribution from the valley $\mathbf{K}$ in the Brillouin zone while $\mathbf{j}_{-\mathbf{K}}(\mathbf{p})$ is the contribution from valley $-\mathbf{K}$. Therefore, Eq. (26) in fact collects contributions from the vicinity of the Dirac points at the opposite corners of the entire Brillouin zone [shaded sectors in Fig. 1(b) or 1(c)].

Using Eq. (23) we obtain from Eq. (5) in the linear approximation,

$$\mathbf{j} = -ev_F\sum_{\alpha,\mathbf{p}}[\hat{u}^{(0)\dagger}_{\alpha}\hat{\boldsymbol{\sigma}}\hat{u}^{(0)}_{\alpha}+\hat{v}^{(0)\dagger}_{\alpha}\hat{\boldsymbol{\sigma}}\hat{v}^{(0)}_{\alpha}][1-2f(E^{(0)}_{\alpha}+E^{(1)}_{\alpha})]$$
$$-2ev_F\,\mathrm{Re}\sum_{\alpha\neq\beta,\mathbf{p}}B_{\alpha\beta}[\hat{u}^{(0)\dagger}_{\alpha}\hat{\boldsymbol{\sigma}}\hat{u}^{(0)}_{\beta}+\hat{v}^{(0)\dagger}_{\alpha}\hat{\boldsymbol{\sigma}}\hat{v}^{(0)}_{\beta}][1-2f(E^{(0)}_{\alpha})]. \tag{29}$$

Equation (29) contains the terms which diverge for large $v_Fp \gg |\Delta|$, T because of the contributions from the Fermi sea of the states with negative energies, which extend over the entire Brillouin zone including regions far from the Dirac point. This divergence is spurious and can be eliminated using two equivalent methods.

First, we note that the divergence of this kind is caused simply by the fact that the overall shift of the particle momentum in the Brillouin zone creates corrections to the wave functions which do not decay as functions of the momentum far from the Dirac points. Let us consider a change in the particle momentum $\mathbf{p}\to\mathbf{p}+\delta\mathbf{p}$ everywhere in the Brillouin zone. It will lead to the shift $\mathbf{p}\to\mathbf{p}+\delta\mathbf{p}$ in the functions u and, at the same time, to the shift $\mathbf{p}\to\mathbf{p}-\delta\mathbf{p}$ in v, because the functions v are associated with the complex conjugated wave functions u taken at the point $-\mathbf{K}$. In this way, the wave functions used in Eq. (7) contain corrections associated with the overall shift $\mathbf{p}\to\mathbf{p}+\mathbf{k}/2$ in the Brillouin zone. It is thus legitimate to simultaneously change the momentum under the integral in Eq. (26) or (29) back to its original value $\mathbf{p}$, i.e., to change the blind integration variable $\mathbf{p}\to\mathbf{p}-\mathbf{k}/2$ in the first and $\mathbf{p}\to\mathbf{p}+\mathbf{k}/2$ in second term. Excluding this momentum shift we thus remove the diverging part, which is not relevant to the supercurrent. Within the linear approximation, it is sufficient to shift the momenta in the zero-order term which comes from the first line of Eq. (29). The zero-order term yields

$$\mathbf{j}^{(0)} = \int\frac{d^2p}{(2\pi)^2}[\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p}-\mathbf{k}/2)+\mathbf{j}^{(0)}_{-\mathbf{K}}(\mathbf{p}+\mathbf{k}/2)]$$
$$=\int\frac{d^2p}{(2\pi)^2}\left[\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})+\mathbf{j}^{(0)}_{-\mathbf{K}}(\mathbf{p})-\left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{p}}\right)\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})\right].$$

Here $\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})$ and $\mathbf{j}^{(0)}_{-\mathbf{K}}(\mathbf{p})$ are the currents, Eqs. (27) and (28), within the zero-order approximation in $\mathbf{k}$, i.e., with the functions $\hat{u}^{(0)}_{\alpha}$ and $\hat{v}^{(0)}_{\alpha}$ and the energies $E^{(0)}_{\alpha}$ of states Eqs. (12)–(17) without a current. For these states $\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})+\mathbf{j}^{(0)}_{-\mathbf{K}}(\mathbf{p})=0$. As a result

$$\mathbf{j}^{(0)} = -\int\frac{d\phi}{(2\pi)^2}[(\mathbf{p}\cdot\mathbf{k})\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})]_{p\gg\Delta,T}. \tag{30}$$

Here we transformed into the surface integral over a remote sphere in the momentum space. At these momenta and energies, the current in Eq. (30) does not contain any information on the superconducting properties of the material. This term compensates the divergence of the corrections in the second line of that equation. This method of regularizing the current is similar to that used in Ref. 11.

Let us now discuss another way how to remove the divergence in the second line of Eq. (29). To do this one can subtract from Eq. (29) the expression for the normal-state current which is identically zero for the wave functions specified by Eq. (7). Indeed, as we already mentioned, the diverging contributions to the current come from the regions far from the Dirac point. Since, at these quasiparticle momenta the energies greatly exceed the scales relevant to the superconducting state, the corresponding contributions to the current coincide with those in the normal state. One can show that this regularization procedure leads to the same result as Eq. (30). The details of calculations are given in Appendix A. This way of getting rid of the spurious divergences of supercurrent was also considered in Ref. 8 (see also discussion in Refs. 16 and 17). It is worthwhile to note that the problem of spurious divergent terms in the expression for the supercurrent is rather general. In particular, it was discussed (and resolved similarly) for the superfluid excitonic current in graphene bilayers.[20]

The final expression for the current becomes

$$\mathbf{j} = ev_F^2\mathbf{k}\int_0^{\infty}\frac{pdp}{2\pi}\left[-\frac{1}{4T}\cosh^{-2}\frac{E_{\uparrow}}{2T}-\frac{1}{4T}\cosh^{-2}\frac{E_{\downarrow}}{2T}\right.$$
$$+\frac{|u^*_{\uparrow}u_{\downarrow}+v^*_{\uparrow}v_{\downarrow}|^2}{E_{\uparrow}-E_{\downarrow}}\left(\tanh\frac{E_{\uparrow}}{2T}-\tanh\frac{E_{\downarrow}}{2T}\right)+\frac{1}{v_Fp}$$
$$\left.-\frac{|v^*_{\uparrow}u_{\downarrow}-u^*_{\uparrow}v_{\downarrow}|^2}{E_{\uparrow}+E_{\downarrow}}\left(\tanh\frac{E_{\uparrow}}{2T}+\tanh\frac{E_{\downarrow}}{2T}\right)\right]. \tag{31}$$

We evaluate Eq. (31) for low temperatures, $T\ll|\Delta|$. Since $E_{\uparrow},E_{\downarrow}>|\Delta|$, the first three terms in Eq. (31) vanish at $T=0$. The supercurrent becomes

$$\mathbf{j} = \frac{e\mathbf{k}}{2\pi}\left[\sqrt{\mu^2+|\Delta|^2}+\frac{|\Delta|^2}{|\mu|}\ln\left(\frac{|\mu|+\sqrt{\mu^2+|\Delta|^2}}{|\Delta|}\right)\right]. \tag{32}$$

This is the central result of our paper. It determines the supercurrent as a linear response to the condensate momentum, i.e., to the gradient of the order-parameter phase, $\mathbf{k}$.

For $\mu\gg|\Delta|$ the supercurrent becomes

$$\mathbf{j} = e|\mu|\mathbf{k}/2\pi. \tag{33}$$

For $\mu\ll|\Delta|$ we find

$$\mathbf{j} = e|\Delta|\mathbf{k}/\pi. \tag{34}$$

This result formally holds within the linear approximation which assumes $v_Fk\ll\mu$. Therefore, in order to receive Eq. (34) one should have to put $k\to 0$ first and then assume $\mu\ll|\Delta|$. In the next section we demonstrate that Eqs. (32)

and (34) are in fact valid as long as $|\mu|\ll|\Delta|$ and $v_F k\ll|\Delta|$ irrespective of the relation between $v_F k$ and μ.

B. Low-doping limit

Consider now the limit of small μ when the zero-order state is degenerate because $E_1^{(0)}=E_3^{(0)}=E_0$ and $E_2^{(0)}=E_4^{(0)}=-E_0$, where

$$E_0=\sqrt{(v_F p)^2+|\Delta|^2}. \quad (35)$$

In what follows we demonstrate that despite the absence of the Doppler term in its usual form, there still is a finite linear in $\mathbf{k}$ supercurrent down to zero temperature (contrary to the result of Ref. 8).

The true wave functions satisfy

$$\check{H}\check{\psi}_\alpha=E_\alpha\check{\psi}_\alpha,$$

where $\check{H}=\check{H}^{(0)}+\check{H}^{(1)}$ and

$$\check{H}^{(0)}=\begin{pmatrix} v_F\hat{\boldsymbol{\sigma}}\cdot\mathbf{p} & |\Delta| \\ |\Delta| & -v_F\hat{\boldsymbol{\sigma}}\cdot\mathbf{p}\end{pmatrix},$$

$$\check{H}^{(1)}=\begin{pmatrix} \frac{1}{2}v_F\hat{\boldsymbol{\sigma}}\cdot\mathbf{k}-\mu & 0 \\ 0 & \frac{1}{2}v_F\hat{\boldsymbol{\sigma}}\cdot\mathbf{k}+\mu\end{pmatrix}.$$

We assume $v_F k,\ \mu\ll E_0$.

Let us expand the true wave function into the zero-order orthonormal wave functions $\check{\psi}_\alpha^{(0)}$,

$$\check{\psi}_\alpha=\sum_\beta C_{\alpha\beta}\check{\psi}_\beta^{(0)}, \quad (36)$$

satisfying the zero-order equation,

$$\check{H}^{(0)}\check{\psi}_\alpha^{(0)}=E_\alpha^{(0)}\check{\psi}_\alpha^{(0)}.$$

The zero-order wave functions have now the form of Eqs. (13) and (15) with $u_\uparrow=v_\downarrow\equiv u$ and $u_\downarrow=v_\uparrow\equiv v$, where

$$u=\frac{1}{\sqrt{2}}\left[1+\frac{v_F p}{E_0}\right]^{1/2},\quad v=\frac{1}{\sqrt{2}}\left[1-\frac{v_F p}{E_0}\right]^{1/2}. \quad (37)$$

The expansion coefficients satisfy

$$C_{\alpha\gamma}[E_\alpha-E_\gamma^{(0)}]=\sum_\beta C_{\alpha\beta}H_{\gamma\beta}, \quad (38)$$

where $H_{\gamma\beta}\equiv\langle\check{\psi}_\gamma^{(0)+}\check{H}^{(1)}\check{\psi}_\beta^{(0)}\rangle$.

Consider the state $\alpha=1$. Since the difference $E_1-E_1^{(0)}=E_1-E_3^{(0)}\equiv\delta E_1$ in Eq. (38) is small, the coefficients C_{12} and C_{14} are proportional to the perturbation while C_{11} and C_{13} are of the order unity. The coefficients $C_{11}^{(0)}$, $C_{13}^{(0)}$ in the leading approximation satisfy the secular Eqs. (B5) and (B6) (see Appendix B) which yield $\delta E_{1,3}=\mp\widetilde{E}_1$,

$$\widetilde{E}_1=\sqrt{\left(\frac{\mu v_F p}{E_0}-E_D\right)^2+\frac{v_F^2[\mathbf{p}\times\mathbf{k}]^2}{4p^2}\frac{|\Delta|^2}{E_0^2}} \quad (39)$$

and

$$C_{11}^{(0)}=C_{33}^{(0)}=\frac{1}{\sqrt{2}}\sqrt{1+\frac{\frac{\mu v_F p}{E_0}-E_D}{\widetilde{E}_1}}, \quad (40)$$

$$C_{13}^{(0)}=C_{31}^{(0)}=\frac{i\,\mathrm{sign}([\mathbf{p}\times\mathbf{k}]\mathbf{z})}{\sqrt{2}}\sqrt{1-\frac{\frac{\mu v_F p}{E_0}-E_D}{\widetilde{E}_1}}. \quad (41)$$

The coefficients obey the normalization $|C_{11}^{(0)}|^2+|C_{13}^{(0)}|^2=|C_{33}^{(0)}|^2+|C_{31}^{(0)}|^2=1$. In the same way we find $\delta E_{2,4}=\pm\widetilde{E}_2$. The energy correction $\widetilde{E}_2$ and the coefficients $C_{22}^{(0)}=C_{44}^{(0)}$ and $C_{24}^{(0)}=C_{42}^{(0)}$ are obtained from $\widetilde{E}_1$, $C_{11}^{(0)}$, and $C_{13}^{(0)}$, respectively, by replacing $\mathbf{p}\to-\mathbf{p}$.

Equations (39)–(41) in the limit $\mu=0$ yield

$$\widetilde{E}_1=\widetilde{E}_2\equiv\widetilde{E}=(v_F p/E_0)\sqrt{E_D^2+k^2|\Delta|^2/4p^2},$$

which agrees with Eq. (21). For $k/p\ll\mu/|\Delta|$ we have

$$\widetilde{E}_{1,2}=\frac{\mu v_F p}{E_0}\mp E_D$$

and $C_{11}=C_{33}=C_{22}=C_{44}=1$, $C_{13}=C_{31}=C_{24}=C_{42}=0$ which agrees with the result of the linear approximation in k. The coefficients C_{ik} taken for $v_F k\ll\mu$ coincide with Eqs. (24) and (25) in the limit $\mu\ll E$.

Now we insert all four eigenstates with the corresponding coefficients $C_{\alpha\beta}$ found from Eq. (38) for a small finite μ into the expression, Eq. (5), for the current. Removing the divergence by subtracting the current in the normal state we find

$$\mathbf{j}=ev_F^2\mathbf{k}\int_0^\infty\frac{pdp}{2\pi}\left[-\left(1+\frac{|\Delta|^2}{E_0^2}\right)\frac{d}{dE_0}\tanh\frac{E_0}{2T}+\frac{1}{v_F p}-\frac{v_F^2p^2}{E_0^3}\tanh\frac{E_0}{2T}\right]. \quad (42)$$

As one sees, it does not depend on μ though the magnitude of $\mu\ll|\Delta|$ was initially a finite quantity taken in an arbitrary proportion to the magnitude of $v_F k$.

Consider low temperatures, $T\ll|\Delta|$. The first term vanishes and we obtain

$$\mathbf{j}=ev_F^2\mathbf{k}\int_0^\infty\frac{pdp}{2\pi}\left[\frac{1}{v_F p}-\frac{v_F^2p^2}{E_0^3}\right]=\frac{e|\Delta|\mathbf{k}}{\pi}, \quad (43)$$

which is the same result as that obtained in the linear-in-$\mathbf{k}$ approximation, Eq. (34). This proves that Eq. (34) holds for $v_F k\ll|\Delta|$ and $\mu\ll|\Delta|$ irrespective of the relation between $v_F k$ and μ.

IV. DISCUSSION AND COMPARISON

As we already mentioned in Sec. I, the present paper studies the model[8,13] that assumes Cooper pairing between electrons (holes) belonging to the same sublattice in the configurational space. This is evident, in fact, from the self-consistency equation, Eq. (3), which contains the scalar product of the spinors $\hat{u}$ and $\hat{v}$. However, other scenarios of the superconducting pairing are possible, as well. In particular, one can use the approach which is based on the Landau Fermi-liquid theory which operates with the quasiparticles corresponding to the eigenstates of the normal-state Hamiltonian [spinors $\hat{a}_\uparrow$ and $\hat{a}_\downarrow$ in Eq. (11)]. All essential properties are then derived based on the quasiparticle energy spectrum. The Fermi-liquid approach (with some variations) was used in Refs. 7 and 9–12. The dilemma of "intrasublattice interaction only" vs "interaction between true quasiparticles of the normal graphene" was also discussed in connection with other collective modes in graphene.[21] In general, this dilemma can be resolved only on the basis of the detailed microscopic analysis of the particular interaction mechanism.

Nevertheless, the main outcome of our analysis is that the superconducting behaviors calculated within the two aforementioned approaches are qualitatively very similar, though intrasublattice interaction requires a more involved algebra (fourfold matrices rather than twofold matrices in the Fermi-liquid approach). Quantitatively, however, Eq. (32) is slightly different from the corresponding Eq. (14) of Ref. 11. In particular, the current in the limit $\mu \gg |\Delta|$, Eq. (33), is twice as large as in Ref. 11. This factor 2 appears simply because the model with two times smaller number of the degrees of freedom (only one valley with a Dirac cone in the Brillouin zone) has been considered in the cited paper. For the limit of large μ, there should otherwise be no difference in the superconducting properties, since the both models practically coincide with that for usual superconductors. Taking into account this twice larger number of degrees of freedom one still notices, however, that the low-μ limit, Eq. (34), has yet an extra factor 2 as compared to that of Ref. 11. This is obviously a manifestation of a more subtle difference between these two models caused by an interplay of the eigenstates near the degeneracy point.

Though the functional dependences in the two models do not exactly coincide, the global features of the superconducting graphene, as functions of the physical parameters, are insensitive to the choice of the pairing model. Moreover, the main message of Ref. 11 is confirmed that the supercurrent and the superconducting electron density are finite at any doping level for all temperatures below the critical temperature; in particular, they do not disappear in the limit $\mu=0$ contrary to the claim of Ref. 8 (this mistake of Ref. 8 has been later admitted by the authors of that paper in Ref. 16).

We have in fact shown that the low-doping limit $\mu \to 0$, being degenerate in the excitation energies, is not any special in the sense of the supercurrent: the supercurrent obtained within the linear approximation in the gradient of the order-parameter phase, $k \ll |\Delta|/v_F$, is the same irrespective of the relation between $v_F k$ and μ. The crucial difference between the superconducting graphene and the usual BCS superconductor is that the supercurrent density in the low-doping limit at $T=0$ is proportional to the order parameter Δ rather than to the total electron density.

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APPENDIX A: CURRENT IN THE LINEAR APPROXIMATION

We start with Eq. (29). Using Eqs. (13), (15), (24), and (25) we find after averaging over momentum directions

$$\mathbf{j} = -ev_F \sum_{\mathbf{p}} \frac{\mathbf{p}}{p}\left[\tanh\frac{E_\uparrow + E_D}{2T} - \tanh\frac{E_\uparrow - E_D}{2T} + \tanh\frac{E_\downarrow + E_D}{2T} - \tanh\frac{E_\downarrow - E_D}{2T}\right] - ev_F^2 \mathbf{k}\sum_{\mathbf{p}}\left[\frac{|u_\uparrow^* u_\downarrow + v_\uparrow^* v_\downarrow|^2}{E_\uparrow - E_\downarrow}\left(\tanh\frac{E_\uparrow}{2T} - \tanh\frac{E_\downarrow}{2T}\right) + \frac{|v_\uparrow^* u_\downarrow - u_\uparrow^* v_\downarrow|^2}{E_\uparrow + E_\downarrow}\left(\tanh\frac{E_\uparrow}{2T} + \tanh\frac{E_\downarrow}{2T}\right)\right]. \quad \text{(A1)}$$

This expression formally diverges for large $v_F p$ due to contribution from regions far from the Dirac points. As we discussed already in Sec. III, we remove this spurious divergence by transforming the zero-order terms. With Eqs. (16) and (17) we have in the zero-order approximation,

$$\mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p}) = -\mathbf{j}_{-\mathbf{K}}^{(0)}(\mathbf{p}) = -ev_F\left[\frac{v_F p - \mu}{E_\uparrow}\tanh\frac{E_\uparrow}{2T} + \frac{v_F p + \mu}{E_\downarrow}\tanh\frac{E_\downarrow}{2T}\right]\frac{\mathbf{p}}{p}. \quad \text{(A2)}$$

The contribution from the zero-order term has the form of Eq. (30) of surface integral over a remote sphere in the momentum space. Using Eq. (A2) we find

$$\mathbf{j}^{(0)} = -\int \frac{d\phi}{(2\pi)^2}[(\mathbf{p}\cdot\mathbf{k})\mathbf{j}_{\mathbf{K}}^{(0)}(\mathbf{p})]_{p\to\infty} = ev_F^2\mathbf{k}\int_0^\infty \frac{pdp}{2\pi}\left[\frac{1}{v_F p}\right]. \quad \text{(A3)}$$

When added to Eq. (A1), this compensates the diverging terms there. As a result, we obtain the converging expression, Eq. (31). The same result can be obtained if we subtract the normal current, i.e., Eq. (A1) for $\Delta=0$.

For $T \ll |\Delta|$ the terms $\cosh^{-2}(E_{\uparrow,\downarrow}/2T)$ are small while $\tanh(E_{\uparrow,\downarrow}/2T)=1$. Therefore, the first three terms in Eq. (31) vanish, and the current becomes

$$\mathbf{j} = ev_F^2\mathbf{k}\int_0^\infty \frac{pdp}{2\pi}\left[\frac{1}{v_F p} - \frac{2|v_\uparrow^* u_\downarrow - u_\uparrow^* v_\downarrow|^2}{E_\uparrow + E_\downarrow}\right]. \quad \text{(A4)}$$

Calculating the current with the help of Eqs. (16) and (17) we obtain Eq. (32).

APPENDIX B: CURRENT IN THE DEGENERATE CASE

1. Wave functions for weak doping $\mu \ll \Delta$

Consider Eq. (38) for the state $\alpha=1$. We have within the linear approximation in $H^{(1)}$,

$$C_{11}[E_1 - E_1^{(0)}] = \sum_\beta C_{1\beta}H_{1\beta}, \tag{B1}$$

$$C_{13}[E_1 - E_3^{(0)}] = \sum_\beta C_{1\beta}H_{3\beta} \tag{B2}$$

and

$$C_{12}[E_1 - E_2^{(0)}] = \sum_\beta C_{1\beta}H_{2\beta}, \tag{B3}$$

$$C_{14}[E_1 - E_4^{(0)}] = \sum_\beta C_{1\beta}H_{4\beta}. \tag{B4}$$

Since

$$E_1 - E_1^{(0)} = E_1 - E_3^{(0)} \equiv \delta E_1$$

are small, in Eqs. (B1) and (B2) we can take the coefficients in the zero-order approximation. As a result C_{12} and C_{14} are small (i.e., are proportional to the perturbation) while C_{11} and C_{13} are of the order unity. We put $C_{11}=C_{11}^{(0)}$, $C_{13}=C_{13}^{(0)}$ while $C_{12}=C_{12}^{(1)}$, $C_{14}=C_{14}^{(1)}$, and find up to the first-order terms,

$$C_{11}^{(0)}\delta E_1^{(1)} = C_{11}^{(0)}H_{11} + C_{13}^{(0)}H_{13}, \tag{B5}$$

$$C_{13}^{(0)}\delta E_1^{(1)} = C_{11}^{(0)}H_{31} + C_{13}^{(0)}H_{33} \tag{B6}$$

while

$$2E_0 C_{12}^{(1)} = C_{11}^{(0)}H_{21} + C_{13}^{(0)}H_{23}, \tag{B7}$$

$$2E_0 C_{14}^{(1)} = C_{11}^{(0)}H_{41} + C_{13}^{(0)}H_{43}. \tag{B8}$$

The similar equations are obtained for the other state $\alpha=3$ which belongs to the same energy E_0.

We have

$$H_{\alpha\beta} = \frac{v_F}{2}(\hat{u}_\alpha^{+(0)}\hat{\boldsymbol{\sigma}}\cdot\mathbf{k}\hat{u}_\beta^{(0)} + \hat{v}_\alpha^{+(0)}\hat{\boldsymbol{\sigma}}\cdot\mathbf{k}\hat{v}_\beta^{(0)}) - \mu(\hat{u}_\alpha^{+(0)}\hat{u}_\beta^{(0)} - \hat{v}_\alpha^{+(0)}\hat{v}_\beta^{(0)}).$$

Therefore

$$H_{11} = -H_{33} = \frac{v_F\mathbf{p}\cdot\mathbf{k}}{2p} - \frac{\mu v_F p}{E_0},$$

$$H_{13} = -H_{31} = \frac{i v_F([\mathbf{z}_0\times\mathbf{p}]\mathbf{k})}{2p}\frac{|\Delta|}{E_0}$$

and

$$H_{21} = H_{21} = H_{43} = H_{34} = -\mu\frac{|\Delta|}{E_0}, \tag{B9}$$

$$H_{23} = -H_{32} = -i\frac{v_F p}{E_0}\frac{v_F([\mathbf{p}\times\mathbf{k}]\mathbf{z})}{2p}, \tag{B10}$$

$$H_{41} = -H_{14} = -i\frac{v_F p}{E_0}\frac{v_F([\mathbf{p}\times\mathbf{k}]\mathbf{z})}{2p}. \tag{B11}$$

Secular Eqs. (B5) and (B6) determine $\delta E_{1,3}$ together with the coefficients $C_{11}^{(0)}$ and $C_{13}^{(0)}$. The first-order corrections to C_{11} and C_{13} are found from Eqs. (B1) and (B2) written up to the second-order terms. Using Eqs. (B9)–(B11) we obtain $C_{11}^{(1)}=C_{13}^{(1)}=0$. The coefficients $C_{12}^{(1)}$ and $C_{14}^{(1)}$ are determined by Eqs. (B7) and (B8). In the same way we find $C_{31}^{(1)}=C_{33}^{(1)}=0$ together with the coefficients $C_{32}^{(1)}$ and $C_{34}^{(1)}$. Calculations for the states $\alpha=2,4$ can be done in exactly the same way.

Using the obtained coefficients we can rewrite Eq. (5) for the current in the form

$$\mathbf{j} = -ev_F\sum_{\mathbf{p}}\left[A_1^{(0)}\left(\tanh\frac{E_0-\tilde{E}_1}{2T} - \tanh\frac{E_0+\tilde{E}_1}{2T}\right) - A_2^{(0)}\times\left(\tanh\frac{E_0-\tilde{E}_2}{2T} - \tanh\frac{E_0+\tilde{E}_2}{2T}\right) + 2(A_1^{(1)} - A_2^{(1)})\tanh\frac{E_0}{2T}\right], \tag{B12}$$

where

$$A_1^{(0)} = (|C_{11}^{(0)}|^2 - |C_{13}^{(0)}|^2)\hat{a}_\uparrow^\dagger\hat{\boldsymbol{\sigma}}\hat{a}_\uparrow + 2uv(C_{11}^{(0)*}C_{13}^{(0)} - C_{13}^{(0)*}C_{11}^{(0)})\hat{a}_\uparrow^\dagger\hat{\boldsymbol{\sigma}}\hat{a}_\downarrow,$$

$$A_2^{(0)} = (|C_{22}^{(0)}|^2 - |C_{24}^{(0)}|^2)\hat{a}_\uparrow^\dagger\hat{\boldsymbol{\sigma}}\hat{a}_\uparrow + 2uv(C_{22}^{(0)*}C_{24}^{(0)} - C_{24}^{(0)*}C_{22}^{(0)})\hat{a}_\uparrow^\dagger\hat{\boldsymbol{\sigma}}\hat{a}_\downarrow$$

and

$$A_1^{(1)} = (u^2 - v^2)E_0^{-1}H_{23}\hat{a}_\uparrow^\dagger\hat{\boldsymbol{\sigma}}\hat{a}_\downarrow,$$

$$A_2^{(1)} = (u^2 - v^2)E_0^{-1}H_{14}\hat{a}_\uparrow^\dagger\hat{\boldsymbol{\sigma}}\hat{a}_\downarrow.$$

The current in Eq. (B12) takes the form

$$\mathbf{j} = 2e\sum_{\mathbf{p}}\left[\frac{\partial\tilde{E}_1}{\partial\mathbf{k}}\left(\tanh\frac{E_0-\tilde{E}_1}{2T} - \tanh\frac{E_0+\tilde{E}_1}{2T}\right) + \frac{\partial\tilde{E}_2}{\partial\mathbf{k}}\left(\tanh\frac{E_0-\tilde{E}_2}{2T} - \tanh\frac{E_0+\tilde{E}_2}{2T}\right) - \frac{v_F^2p^2}{E_0^3}\frac{v_F^2[\mathbf{p}\times\mathbf{k}]\times\mathbf{p}}{p^2}\tanh\frac{E_0}{2T}\right]. \tag{B13}$$

For $v_Fk \ll \mu$ this equation coincides with Eq. (A1) taken for $\mu \ll \Delta$. Expanding it in small $\tilde{E}_1$ and $\tilde{E}_2$ we find

$$\mathbf{j} = -2e\sum_{\mathbf{p}}\left[\frac{\partial}{\partial\mathbf{k}}(\tilde{E}_1^2 + \tilde{E}_2^2)\frac{d}{dE_0}\tanh\frac{E_0}{2T} + \frac{v_F^2p^2}{E_0^3}\frac{v_F^2[\mathbf{p}\times\mathbf{k}]\times\mathbf{p}}{p^2}\tanh\frac{E_0}{2T}\right],$$

which yields after integration over the momentum angle,

$$\mathbf{j} = -ev_F^2\mathbf{k}\sum_{\mathbf{p}}\left[\left(1+\frac{|\Delta|^2}{E_0^2}\right)\frac{d}{dE_0}\tanh\frac{E_0}{2T}+\frac{v_F^2p^2}{E_0^3}\tanh\frac{E_0}{2T}\right]. \tag{B14}$$

Subtraction of the normal-state current returns us to Eq. (42).

2. Undoped graphene

The Bogoliubov-de Gennes equations can be solved exactly for the undoped case $\mu=0$ where the dispersion Eq. (20) becomes biquadratic with the energy spectrum, Eq. (21). We shall write down the solution of the Bogoliubov-de Gennes equations for the vector $\mathbf{k}$ parallel to the axis x ($k=k_x$). Then

$$E_\pm^2 = (\sqrt{|\Delta|^2+v_F^2p_x^2} \pm v_Fk/2)^2 + v_F^2p_y^2 \tag{B15}$$

and one may check by substitution that the four orthonormalized solutions of the Bogoliubov-de Gennes equations are given by the spinors,

$$\hat{u} = \frac{1}{2}\sqrt{1 \pm \frac{p_x}{\sqrt{|\Delta|^2/v_F^2+p_x^2}}}\begin{pmatrix} \pm\dfrac{E_\pm}{|E_\pm|}e^{-i\phi_\pm/2} \\ e^{i\phi_\pm/2}\end{pmatrix},$$

$$\hat{v} = \frac{1}{2}\sqrt{1 \mp \frac{p_x}{\sqrt{|\Delta|^2/v_F^2+p_x^2}}}\begin{pmatrix} \pm e^{-i\phi_\pm/2} \\ \dfrac{E_\pm}{|E_\pm|}e^{i\phi_\pm/2}\end{pmatrix}, \tag{B16}$$

where the phase factors are given by

$$e^{i\phi_\pm} = \frac{\sqrt{|\Delta|^2/v_F^2+p_x^2} \pm k/2 \pm ip_y}{\sqrt{(\sqrt{|\Delta|^2/v_F^2+p_x^2} \pm k/2)^2+p_y^2}}.$$

Using Eq. (B16) one finds the terms which determine the x component of the supercurrent,

$$\hat{u}^\dagger\hat{\sigma}_x\hat{u} = \pm\frac{1}{2}\frac{E_\pm}{|E_\pm|}\sqrt{1 \pm \frac{p_x}{\sqrt{|\Delta|^2/v_F^2+p_x^2}}}\cos\phi_\pm \approx \pm\frac{1}{2}\frac{E_\pm}{|E_\pm|}\sqrt{1 \pm \frac{p_x}{\sqrt{|\Delta|^2/v_F^2+p_x^2}}}\times\left(1 \pm \frac{kp_y^2}{(|\Delta|^2/v_F^2+p^2)^{3/2}}\right),$$

$$\hat{v}^\dagger\hat{\sigma}_x\hat{v} = \pm\frac{1}{2}\frac{E_\pm}{|E_\pm|}\sqrt{1 \mp \frac{p_x}{\sqrt{|\Delta|^2/v_F^2+p_x^2}}}\cos\phi_\pm \approx \pm\frac{1}{2}\frac{E_\pm}{|E_\pm|}\sqrt{1 \mp \frac{p_x}{\sqrt{|\Delta|^2/v_F^2+p_x^2}}}\times\left(1 \pm \frac{kp_y^2}{(|\Delta|^2/v_F^2+p^2)^{3/2}}\right),$$

where finally we keep only terms linear in k. Collecting all the terms in the expression, Eq. (4), we arrive at Eq. (34).

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